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Resonant tunnelling through short-range singular potentials

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Abstract
A three-parameter family of point interactions constructed from sequences of symmetric barrier–well–barrier and well–barrier–well rectangles is studied in the limit, when the rectangles are squeezed to zero width but the barrier height and the well depth become infinite (the zero-range limit). The limiting generalized potentials are referred to as the second derivative of Dirac's delta function $\pm\lambda\delta''(x)$ with a renormalized coupling constant $\lambda > 0$ or simply as $\pm\delta''$-like point interactions. As a result, a whole family of self-adjoint extensions of the one-dimensional Schrödinger operator is shown to exist, which results in full and partial resonant tunnelling through this class of singular potentials. The resonant tunnelling occurs for countable sets of interaction strength values in the λ-space which are the roots of several transcendental equations. The comparison with the previous results for δ'-like point interactions is also discussed.

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1. Introduction

Point and contact interactions are widely used in various areas of quantum physics (see [1, 2] and references therein, including a large number of other applications, e.g., [3–8]). Intuitively, these interactions are understood as sharply localized potentials, exhibiting a number of interesting and intriguing features [9–24]. Applications of these models to condensed matter physics (see, e.g., [25–29]) are of particular interest nowadays, mainly because of the rapid progress in fabricating nanoscale quantum devices. Other applications arise in optics, for instance, in dielectric media where electromagnetic waves scatter at boundaries or thin layers [30].

In the following we use the quantum-mechanical terminology and consider the limit which neglects the interactions between electrons. In this case, the one-dimensional Schrödinger equation with a potential $V(x)$ for a stationary state reads

$$-\psi''(x) + V(x)\psi(x) = E\psi(x), \tag{1}$$

where the prime stands for the differentiation with respect to the spatial coordinate x, $\psi(x)$ is the wavefunction for a particle of mass m (we use units in which $\hbar^2/2m = 1$) and E is (positive, zero or negative) energy. The present paper deals with the scattering properties of equation (1), where the potential $V(x)$ describes a whole family of point interactions with singularity at $x = 0$. Below they are defined as a zero-range limit of the potentials constructed from rectangular barriers and wells.

A system with a point interaction can be described as a self-adjoint extension of the kinetic energy operator $-\mathrm{d}^2/\mathrm{d}x^2$ with the boundary conditions for the wavefunction $\psi(x)$ and its derivative $\psi'(x)$ connected at the singular point $x = 0$ through a two-by-two transfer matrix Λ [15, 20],

$$\begin{pmatrix}\psi(+0)\\ \psi'(+0)\end{pmatrix} = \Lambda\begin{pmatrix}\psi(-0)\\ \psi'(-0)\end{pmatrix} = \mathrm{e}^{\mathrm{i}\chi}\begin{pmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{21} & \lambda_{22}\end{pmatrix}\begin{pmatrix}\psi(-0)\\ \psi'(-0)\end{pmatrix}, \tag{2}$$

with the real parameters $\chi \in [0, \pi)$, $\lambda_{ij} \in \mathbb{R}$, fulfilling the equation $\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = 1$. However, the connection condition (2) does not describe perfect walls at $x = 0$ through which no probability flow can penetrate, resulting in complete separation between the left and the right half-lines $\mathbb{R}^-$ and $\mathbb{R}^+$. For this reason, the point interactions of this type are called *separated*, in contrast to the *non-separated* point interactions described by the connection condition (2). For separating states, instead of the matrix equation (2), we have the following two conditions [20]:

$$\psi'(-0) = \lambda^-\psi(-0) \qquad \text{and} \qquad \psi'(+0) = \lambda^+\psi(+0), \tag{3}$$

with the parameters $\lambda^\pm \in \mathbb{R} \cup \{\infty\}$. If, e.g., $\lambda^- = \infty$, then the first equation (3) reads $\psi(-0) = 0$ and similarly for $\lambda^+ = \infty$.

On the other hand, instead of using the two connection conditions (2) and (3), for the description of both the non-separated and separated point interactions one can use the approach developed in [21–23], according to which the boundary condition is given in terms of the two-component boundary vectors

$$\Psi \doteq \begin{pmatrix}\psi(+0)\\ \psi(-0)\end{pmatrix} \qquad \text{and} \qquad \Psi' \doteq \begin{pmatrix}\psi'(+0)\\ -\psi'(-0)\end{pmatrix}. \tag{4}$$

This boundary condition reads [21–23, 31]

$$(U - I)\Psi + \mathrm{i}L_0(U + I)\Psi' = 0 \tag{5}$$

with a two-by-two unitary matrix $U \in \mathrm{U}(2)$, the unit matrix I and an arbitrary non-zero constant L_0 of length dimension. A standard parametrization for $U \in \mathrm{U}(2)$ is given by

$$U = \mathrm{e}^{\mathrm{i}\xi}\begin{pmatrix}\alpha & \beta\\ -\beta^* & \alpha^*\end{pmatrix} = \mathrm{e}^{\mathrm{i}\xi}\begin{pmatrix}\alpha_R + \mathrm{i}\alpha_I & \beta_R + \mathrm{i}\beta_I\\ -\beta_R + \mathrm{i}\beta_I & \alpha_R - \mathrm{i}\alpha_I\end{pmatrix}, \tag{6}$$

where $\xi \in [0, \pi)$ and α, β are complex parameters satisfying

$$|\alpha|^2 + |\beta|^2 = \alpha_R^2 + \alpha_I^2 + \beta_R^2 + \beta_I^2 = 1. \tag{7}$$

Note that the description (5) with (6) can be put into the connection form (2) only if $\beta \neq 0$ and, in this case, we have [23, 31]

$$\Lambda = \frac{\mathrm{i}}{\beta_R - \mathrm{i}\beta_I}\begin{pmatrix}\sin\xi - \alpha_I & -L_0(\cos\xi + \alpha_R)\\ L_0^{-1}(\cos\xi - \alpha_R) & \sin\xi + \alpha_I\end{pmatrix}. \tag{8}$$

For $\beta = 0$, the boundary condition (5) splits into two conditions [23, 31] which are the counterparts of (3).

If we suppose the interaction $V(x)$ to be invariant under space reflection $x \to -x$, then the transformation

$$\psi(\pm 0) \to \psi(\mp 0) \qquad \text{and} \qquad \psi'(\pm 0) \to -\psi'(\mp 0) \tag{9}$$

has to keep equation (2) but this occurs if and only if $\lambda_{11} = \lambda_{22}$ and $\chi = 0$.

Thus, in the particular case of the point potential given in the form of Dirac's delta function,

$$V(x) = g\delta(x), \tag{10}$$

with g being a coupling constant, the constant values in (2) become

$$\lambda_{11} = \lambda_{22} = 1, \qquad \lambda_{12} = 0, \qquad \lambda_{21} = g, \qquad \chi = 0. \tag{11}$$

In this case, the type of boundary conditions (2) assumes that the wavefunction $\psi(x)$ is continuous but its first derivative discontinuous at the singularity point $x = 0$. This is a quite simple example of point interactions in one dimension. Because of the continuity of the wavefunction $\psi(x)$, the product $\delta(x)\psi(x)$ becomes well defined at $x = 0$ and therefore the solution of equation (1) with potential (10) is unique. Using equations (2) and (8), instead of conditions (11), we obtain

$$\alpha_I = \beta_R = 0, \qquad \alpha_R = -\cos\xi, \qquad \beta_I = -\sin\xi, \tag{12}$$

where the parameter ξ is given by the equation

$$\tan\xi = 2/L_0 g. \tag{13}$$

There exists a special case of tunnelling through the antisymmetric δ'-potential defined as the derivative of Dirac's delta function in the sense of distributions, i.e.,

$$V(x) = \lambda\delta'(x), \qquad \delta'(x) \doteq \mathrm{d}\delta(x)/\mathrm{d}x, \tag{14}$$

with a coupling constant λ, when both the wavefunction $\psi(x)$ and its derivative $\psi'(x)$ appear to be discontinuous at $x = 0$ [32, 33]. For this case, the matrix equation (2) is invariant under space reflection $x \to -x$, if instead of transformation (9), the substitutions

$$\begin{aligned} &\psi(+0) \to C^{-1}\psi(-0), \qquad \psi(-0) \to C\psi(+0), \\ &\psi'(+0) \to C\psi'(-0), \qquad \psi'(-0) \to C^{-1}\psi'(+0) \end{aligned} \tag{15}$$

with any real constant C are used, but this kind of invariance occurs if and only if

$$\lambda_{11}^{-1} = \lambda_{22} = C, \qquad \lambda_{12} = \lambda_{21} = 0 \qquad \text{and} \qquad \chi = 0. \tag{16}$$

The point interaction of this type becomes non-trivial if $C \neq 1$. In this case, according to equations (2) and (8), the matrix $U \in \mathrm{U}(2)$ is given by

$$\alpha_R = \beta_R = 0, \qquad \alpha_I = \frac{C^2 - 1}{C^2 + 1}, \qquad \beta_I = -\frac{2C}{C^2 + 1}, \qquad \xi = \frac{\pi}{2}. \tag{17}$$

In this paper, we study a family of point interactions with non-trivial scattering properties which exhibit full resonant transparency. Since for the appearance of full transparency the potential $V(x)$ in equation (1) has to be a symmetric function under the transformation $x \to -x$, a fully transparent system with resonant tunnelling can be constructed from two identical barriers. Besides this, in order to have an interaction located at a single point, some kind of singularity in the form of a well located between these barriers has to be incorporated into the double-well system as well. Therefore the zero-range limit of two rectangular barriers separated by a rectangular well, when the barrier height and the well depth tend to infinity as the barriers and the well are squeezed to zero width, can be used as a simple particular choice of full resonant tunnelling because it can analytically be treated yielding explicit solutions. It turns out that the structure with opposite sign, i.e., a barrier surrounded by two identical wells, in the zero-range limit can also admit a fully transparent regime. The sequence of such barrier–well–barrier (BWB) or well–barrier–well (WBW) stepwise functions with an appropriate squeezing parameter can be considered as a finite regularization of the second derivative of Dirac's delta function used for the point potential

$$V(x) = \pm\lambda\delta''(x), \qquad \delta''(x) \doteq \mathrm{d}^2\delta(x)/\mathrm{d}x^2, \tag{18}$$

with a coupling constant $\lambda > 0$. Similarly to [9, 33], in the present paper we consider a more general situation when potential (18) has a renormalized constant. To this end, we introduce into the regularization scheme of potential (18) three parameters and refer to such potentials in the zero-range limit as to $\pm\delta''$-like point potentials. Using this scheme, the same as in [33], we show that it is possible to define a whole class of $\pm\delta''$-like point interactions with full resonant transparency. As a result, the regions of renormalization parameters for the δ'- and $\pm\delta''$-interactions appear to be same, but the scattering matrix and conditions for resonant tunnelling appear to be different.

2. A rectangular BWB regularization of the potential $\lambda\delta''(x)$

For regularization of interaction (18) with positive sign we approximate it by two identical barriers of height h and width l separated by a well of depth d and width $2r$. Hence the regularized potential is assumed to be a symmetric stepwise function defined by

$$V_{l,2r,l}(x) \doteq \lambda \begin{cases} 0 & \text{for} \quad -\infty < x < -l-r \\ h & \text{for} \quad -l-r < x < -r \\ -d & \text{for} \quad -r < x < r \\ h & \text{for} \quad r < x < l+r \\ 0 & \text{for} \quad l+r < x < \infty \end{cases} \tag{19}$$

with arbitrary positive constants h, l, d, r. This BWB system can be considered as the sum of the regularized dipoles $\delta'(x)$ and $-\delta'(x)$ attached each to other at the point $x = 0$.

We are looking for positive-energy solutions of equation (1) with potential (19) in the form

$$\psi(x) = \begin{cases} \mathrm{e}^{\mathrm{i}kx} + R\,\mathrm{e}^{-\mathrm{i}kx} & \text{for} \quad -\infty < x < -l-r \\ A_1\,\mathrm{e}^{px} + B_1\,\mathrm{e}^{-px} & \text{for} \quad -l-r < x < -r \\ A_2\sin(qx) + B_2\cos(qx) & \text{for} \quad -r < x < r \\ A_3\,\mathrm{e}^{px} + B_3\,\mathrm{e}^{-px} & \text{for} \quad r < x < l+r \\ T\,\mathrm{e}^{\mathrm{i}kx} & \text{for} \quad l+r < x < \infty, \end{cases} \tag{20}$$

where R and T are the reflection and transmission amplitudes (from the left), respectively, and

$$k = \sqrt{E}, \qquad p = \sqrt{\lambda h - E} \qquad \text{and} \qquad q = \sqrt{\lambda d + E}. \tag{21}$$

The unknown coefficients A_j and B_j, $j = 1, 2, 3$, are eliminated in a standard way by matching the solutions at the boundaries $x = \pm r$ and $x = \pm(l+r)$. As a result, the scattering amplitudes R and T can be written as follows,

$$R = -\frac{\mathrm{i}W}{\Delta_1 + \mathrm{i}\Delta_2}\,\mathrm{e}^{-2\mathrm{i}k(l+r)} \qquad \text{and} \qquad T = \frac{1}{\Delta_1 + \mathrm{i}\Delta_2}\,\mathrm{e}^{-2\mathrm{i}k(l+r)}, \tag{22}$$

with the following notations:

$$W \doteq \frac{\lambda}{k} D_1 \cosh^2(pl) \cos(2qr), \tag{23}$$

$$D_1 \doteq \frac{h}{p} \tanh(pl) - \frac{fd}{2q} \tan(2qr), \tag{24}$$

$$f \doteq 1 + \left[\left(1 + \frac{h}{d}\right) \frac{k^2}{p^2} - \frac{h}{d}\right] \tanh^2(pl), \tag{25}$$

$$\Delta_1 \doteq \cosh(2pl) \cos(2qr) + \frac{1}{2}\left(\frac{p}{q} - \frac{q}{p}\right) \sinh(2pl) \sin(2qr), \tag{26}$$

$$\Delta_2 \doteq D_2 \cosh^2(pl) \cos(2qr), \tag{27}$$

$$D_2 \doteq \left(\frac{p}{k} - \frac{k}{p}\right) \tanh(pl) - \frac{1}{2}\left(\frac{q}{k} + \frac{k}{q}\right) \tan(2qr) + \frac{1}{2}\left(\frac{p^2}{kq} + \frac{kq}{p^2}\right) \tanh^2(pl) \tan(2qr). \tag{28}$$

The reflectivity and transmissivity can be represented in the form

$$\mathcal{R} \doteq |R|^2 = \frac{W^2}{1 + W^2} \qquad \text{and} \qquad \mathcal{T} \doteq |T|^2 = \frac{1}{1 + W^2}. \tag{29}$$

Therefore the condition of full transparency is $W = 0$ or $D_1 = 0$, and according to equation (24) this condition takes the following explicit form:

$$\frac{h}{p} \tanh(pl) = \frac{fd}{2q} \tan(2qr). \tag{30}$$

In the $d \to 0$ limit, we have $q \to k$ and the last equation takes the well-known form for the double-well structure:

$$2 \cot(2kr) = \left(\frac{k}{p} - \frac{p}{k}\right) \tanh(pl), \tag{31}$$

i.e., equation (30) can be considered as a generalization of the resonant tunnelling condition (31) to the case with two identical barriers separated by a well.

The finite-range expressions given by equations (22)–(28) with the squeezing parameters l and r will be used below to obtain the zero-range limit of potential (19) when $l \to 0$ and $r \to 0$ simultaneously. In this way, we are able to define a whole family of renormalized versions of the point interaction (18) with positive sign.

3. Scattering amplitudes in the zero-range limit

A whole family of δ''-like point interactions can be constructed from squeezing the BWB system (19) if both the barrier height h and the well depth d increase to infinity. Consider the general case

$$h = al^{-\mu} \qquad \text{and} \qquad d = bl^{-\nu} \tag{32}$$

with arbitrary positive constants a, b, μ, ν, where width l serves as a squeezing parameter. As regards the well width r, using the condition that for a δ''-like BWB system, the area above the x-axis and the area below this axis must be equal, we obtain the relation between l and r:

$$r = \eta l^{1-\mu+\nu}, \qquad \eta \doteq a/b. \tag{33}$$

This relation shows how the behaviour of width r depends on the squeezing parameter l. In general, we are interested in the case of single δ''-like point interactions, when the distance r also goes to zero as $l \to 0$. In this case, as follows from equation (33), the inequality $1 - \mu + \nu > 0$ has to be imposed as a necessary condition for obtaining point interactions in the $l \to 0$ limit. Then the set of all single-point interactions being a subset of the quadrant $\{\mu > 0, \nu > 0\}$ appears to be bounded from below by the line $\nu = \mu - 1$ (shown in figure 1 by the dashed line). More precisely, the region of all possible single-point interactions is given by the set

$$\Omega_p \doteq \{\mu > 0, \nu > 0 \mid \mu - 1 < \nu < \infty\}. \tag{34}$$

Using now equations (21), (32) and (33), in the $l \to 0$ limit we find the following asymptotical behaviour for the set Ω_p,

$$p = \sigma l^{-\mu/2}\left(1 - \frac{k^2}{2\sigma^2} l^{\mu} + \cdots\right), \qquad q = \frac{\sigma}{\sqrt{\eta}} l^{-\nu/2}\left(1 + \frac{\eta k^2}{2\sigma^2} l^{\nu} + \cdots\right), \tag{35}$$

for any positive μ and ν. Here we have incorporated the new notation for the coupling constant:

$$\sigma \doteq \sqrt{\lambda a}. \tag{36}$$

In order to find the asymptotics in the $l \to 0$ limit for W, Δ_1 and Δ_2 given by equations (23)–(28), it is convenient to consider separately the following four cases: (i) $pl \to 0$ and $qr \to 0$; (ii) $pl \to 0$ and $qr \to$ const; (iii) $pl \to$ const and $qr \to 0$; (iv) $pl \to$ const and $qr \to$ const.

Case (i): $pl \to 0$ and $qr \to 0$. The region on the quadrant $\{\mu > 0, \nu > 0\}$, where these limits occur simultaneously, can easily be found from expansions (35). Indeed, it follows from the $pl \to 0$ limit that $\mu < 2$, while from the $qr \to 0$ limit we obtain the inequality $2 - 2\mu + \nu > 0$. Both these inequalities define the set

$$\Omega_0 \doteq \{\mu > 0, \nu > 0 \mid 0 < \mu < 2, 2(\mu - 1) < \nu < \infty\} \tag{37}$$

which is a subset of Ω_p (see figure 1). Using expansions (35), for the set Ω_0 we obtain the expansions

$$\tanh(pl) = \sigma l^{1-\mu/2}\left(1 - \frac{\sigma^2}{3} l^{2-\mu} - \frac{k^2}{2\sigma^2} l^{\mu} + \cdots\right), \tag{38}$$

$$\tan(2qr) = 2\sqrt{\eta}\sigma l^{1-\mu+\nu/2}\left(1 + \frac{\eta k^2}{2\sigma^2} l^{\nu} + \frac{4\eta\sigma^2}{3} l^{2-2\mu+\nu} + \cdots\right). \tag{39}$$

Using next expansions (35) and (38) in equation (25), on the set Ω_0, including its boundary $2 - 2\mu + \nu = 0$, we find

$$f = 1 + k^2 l^2 - \eta\sigma^2 l^{2-2\mu+\nu} + \tfrac{2}{3}\eta\sigma^4 l^{4-3\mu+\nu} + \eta k^2 l^{2-\mu+\nu} + \cdots. \tag{40}$$

As a result, using expansions (35), (38)–(40) in equations (24) and (28), we obtain in the $l \to 0$ limit for the set Ω_0 the final asymptotics:

$$\Delta_1 \to 1 \qquad \text{and} \qquad W, \Delta_2 \to -\frac{\sigma^4}{3k}(l^{3-2\mu} + \eta l^{3-3\mu+\nu}). \tag{41}$$

As follows from limits (41), $W, \Delta_2 \to 0$ if both the inequalities $3-2\mu > 0$ and $3(1-\mu)+\nu > 0$ hold simultaneously. Therefore (see equations (22) or (29)) the set

$$\Omega_f \doteq \{\mu > 0, \nu > 0 \mid \mu < 3/2, 3(\mu - 1) < \nu < \infty\}, \tag{42}$$

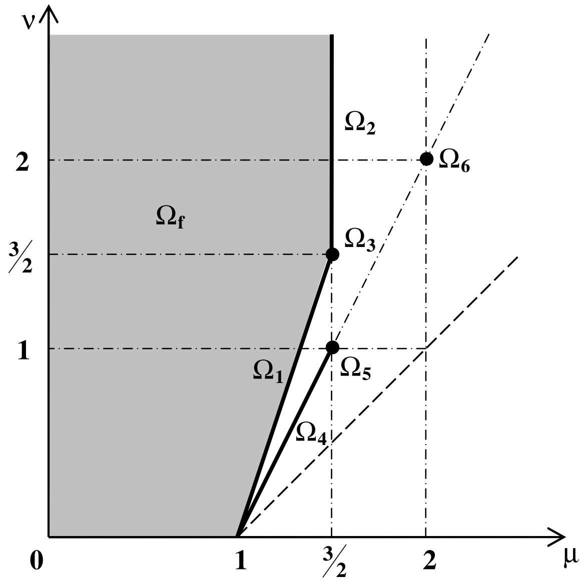

Figure 1. Diagram of existence of δ''-like single-point interactions including the sets of resonances. The shaded set shows the region of fully transparent interactions Ω_f, whereas its boundary $L_1 \doteq \Omega_1 \cup \Omega_2 \cup \Omega_3$ is the set of effective δ-interactions. Resonant tunnelling occurs along the line $L_2 \doteq \Omega_4 \cup \Omega_5$ and at the isolated point Ω_6.

being a subset of Ω_0 (see figure 1), is a region of full transparency for the family of δ''-like point interactions defined through equations (32).

Let us now consider the boundary of the set Ω_f consisting of the two lines $\Omega_1 = \{1 < \mu < 3/2, \nu = 3(\mu - 1)\}$, $\Omega_2 = \{\mu = 3/2, 3/2 < \nu < \infty\}$ and the single point $\Omega_3 = \{\mu = \nu = 3/2\}$, which are shown in figure 1. It follows from asymptotics (41) that except for Δ_1, the quantities W and Δ_2 take also finite values. As a result, according to equation (22), we obtain the same scattering amplitudes as for the δ-interaction (10), namely

$$R = \frac{1}{2\mathrm{i}k/g - 1} \qquad \text{and} \qquad T = \frac{1}{1 + \mathrm{i}g/2k}, \tag{43}$$

but with the renormalized coupling constant given by the equations

$$g = -\frac{2}{3}\sigma^4 \begin{cases} \eta & \text{for} \quad \Omega_1 \\ 1 & \text{for} \quad \Omega_2 \\ 1+\eta & \text{for} \quad \Omega_3. \end{cases} \tag{44}$$

Thus, for all these three sets, i.e., for the line $L_1 \doteq \Omega_1 \cup \Omega_2 \cup \Omega_3$, the transmission through the δ''-like point potential is the same as for the δ-well potential (10) with the coupling constant (44). As expected intuitively, on the line L_1 the effective coupling constant g for the δ''-like interaction is twice bigger than for the corresponding δ'-like interaction [33].

Case (ii): $pl \to 0$ and $qr \to$ const. According to expansions (35), this situation occurs on the line $\nu = 2(\mu - 1)$ where $qr \to \sqrt{\eta}\sigma$ as $l \to 0$. Here asymptotics (40) are slightly modified to

$$f = 1 - \eta\sigma^2 + \tfrac{2}{3}\eta\sigma^4 l^{2-\mu} + \eta k^2 l^{\mu} + \cdots \tag{45}$$

and instead of asymptotics (39) we have to use the expansion

$$\tan(2qr) = \tan(2\sqrt{\eta}\sigma) + \mathcal{O}(l^{2\mu-2}). \tag{46}$$

Then, under the condition

$$(1 - \eta\sigma^2)\tan(2\sqrt{\eta}\sigma) = 2\sqrt{\eta}\sigma, \tag{47}$$

on the line $\nu = 2(\mu - 1)$ we obtain from equations (23), (24), (27) and (28) that in the zero-range limit

$$W, \Delta_2 \to -\frac{\sigma^4}{3k}[\cos(2\sqrt{\eta}\sigma) + \sqrt{\eta}\sigma\sin(2\sqrt{\eta}\sigma)]l^{3-2\mu}. \tag{48}$$

Next, from equation (24) on this line we also find the finite limit

$$\Delta_1 \to \cos(2\sqrt{\eta}\sigma) + \sqrt{\eta}\sigma\sin(2\sqrt{\eta}\sigma) \tag{49}$$

as $l \to 0$ but without constraint (47).

Therefore $W, \Delta_2 \to 0$ on the line $\nu = 2(\mu - 1)$ if $\mu < 3/2$ (due to the presence of the factor $l^{3-2\mu}$ in asymptotics (48)) and this happens for those values of the constant σ which are solutions to equation (47). However, $W, \Delta_2 \to \infty$ for other values of σ, which do not satisfy equation (47). At the limiting point on the line $\nu = 2(\mu - 1)$ where $\mu = 3/2$, both W and Δ_2 become non-zero finite constants. In the following we denote this line and its limiting point by

$$\Omega_4 \doteq \{1 < \mu < 3/2, \nu = 2(\mu - 1)\}, \qquad \Omega_5 \doteq \{\mu = 3/2, \nu = 1\}, \tag{50}$$

respectively (see figure 1). Thus, on the whole line $L_2 \doteq \Omega_4 \cup \Omega_5$, where $pl \to 0$ and $qr \to \sqrt{\eta}\sigma$, we have the limits $W, \Delta_2 \to \infty$, except for those values of the parameter σ which satisfy equation (47).

Equation (47) can be rewritten as a quadratic equation with respect to $\tan(\sqrt{\eta}\sigma)$, leading to the following two equations:

$$\tan(\sqrt{\eta}\sigma) = \sqrt{\eta}\sigma \qquad \text{and} \qquad \tan(\sqrt{\eta}\sigma) = -\frac{1}{\sqrt{\eta}\sigma}. \tag{51}$$

The first equation coincides with the condition for resonant tunnelling through the rectangular barrier–well system [33]. The second equation describes new resonances, which appear due to the presence of the second barrier in the BWB system. As illustrated by figure 2, the set of (non-zero) roots of the first equation $\{\bar{\sigma}_n\}_{n=1}^{\infty}$ is located on the intervals

$$n\pi\eta^{-1/2} < \bar{\sigma}_n < (n + 1/2)\pi\eta^{-1/2}, \qquad n = 1, 2, \ldots, \tag{52}$$

whereas the second series of roots $\{\bar{\tau}_n\}_{n=1}^{\infty}$ lies on the intervals

$$(n - 1/2)\pi\eta^{-1/2} < \bar{\tau}_n < n\pi\eta^{-1/2}, \qquad n = 1, 2, \ldots. \tag{53}$$

Let us denote each pair of the resonance points $\bar{\sigma}_n$ and $\bar{\tau}_n$ by $\bar{s}_n$. Then, using equations (51), we find that at these points

$$\cos(2\sqrt{\eta}\bar{s}_n) + \sqrt{\eta}\bar{s}_n\sin(2\sqrt{\eta}\bar{s}_n) = \pm 1, \qquad n = 1, 2, \ldots, \tag{54}$$

where the upper sign corresponds to the resonances at $\{\bar{\sigma}_n\}_{n=1}^{\infty}$ and the lower one for $\{\bar{\tau}_n\}_{n=1}^{\infty}$. Therefore, due to asymptotics (48) and (49), in the $l \to 0$ limit we find that $\Delta_1 \to \pm 1$ on the whole line L_2, while W and Δ_2 have (equal) non-zero limits only at the limiting point Ω_5, namely, $W, \Delta_2 \to \mp\bar{s}_n^4/3k, n = 1, 2, \ldots$.

According to equations (22), the reflection and transmission amplitudes on the line Ω_4 in the $l \to 0$ limit are given by

$$R \to 0 \qquad \text{and} \qquad T \to \pm 1, \tag{55}$$

while at the point Ω_5 these amplitudes, similarly to (43), also describe the effective δ-interaction. They have almost the same form as amplitudes (43),

$$R = \frac{1}{2\mathrm{i}k/g - 1} \qquad \text{and} \qquad T = \pm\frac{1}{1 + \mathrm{i}g/2k}, \tag{56}$$

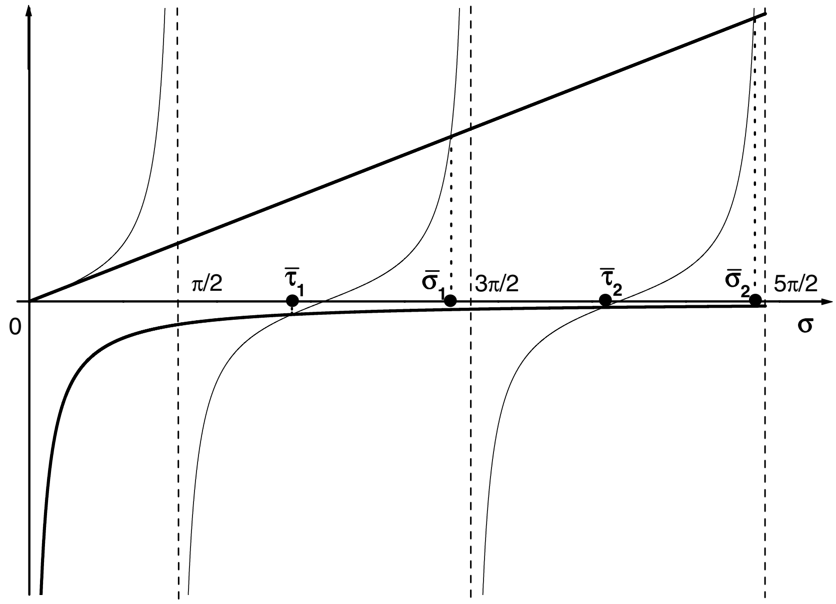

Figure 2. Graphical solutions $\bar{s}_n = (\bar{\sigma}_n, \bar{\tau}_n)$, $n = 1, 2, \ldots$, of equations (51) with $\eta = 1$ in the σ-space shown by dots along the σ-axis.

but, in this case, with the effective coupling constant

$$g = -2\bar{s}_n^4/3, \qquad n = 1, 2, \ldots. \tag{57}$$

The upper sign expression of the second equation (56) corresponds to the resonances with $\sigma = \bar{\sigma}_n$ and the lower one to the resonances with $\sigma = \bar{\tau}_n$, $n = 1, 2, \ldots$.

Thus, at the resonances, which occur in the σ-space at the values $\{\bar{s}_n\}_{n=1}^{\infty}$, the transmission is full on the set Ω_4, while at the point Ω_5, the transmission is effectively the same as for the δ-interaction with the renormalized coupling constant (57).

Case (iii): $pl \to$ const. and $qr \to 0$. It follows from asymptotics (35) that this case, namely $pl \to \sigma$ and $qr \to 0$, corresponds to the vertical line $\{\mu = 2, 2 < \nu < \infty\}$ (see figure 1). On this line, instead of asymptotics (38), we have

$$\tanh(pl) = \tanh\sigma + \mathcal{O}(l^2), \tag{58}$$

while in (39) we should put $\mu = 2$. Using these asymptotics, we find that $f \to 1$ (see equations (25) and (32)) and finally from equation (24) we obtain that $D_1 \to \infty$ as $l \to 0$. Therefore case (iii) deals with full reflection ($R = -1$ and $T = 0$).

Case (iv): $pl \to$ const and $qr \to$ const. Here, due to expansions (35), we have the non-zero finite limits: $pl \to \sigma$ and $qr \to \sqrt{\eta}\sigma$. These limits take place only at the isolated point $\Omega_6 \doteq \{\mu = \nu = 2\}$ (see figure 1). Using asymptotics (46) and (58), for this case we obtain

$$f = 1 - \eta\tanh^2\sigma + \mathcal{O}(l^2). \tag{59}$$

As a result, similarly to case (ii), one obtains that $W, \Delta_2 \to \infty$, except for those values of the constant σ which satisfy the equation

$$(1 - \eta\tanh^2\sigma)\tan(2\sqrt{\eta}\sigma) = 2\sqrt{\eta}\tanh\sigma. \tag{60}$$

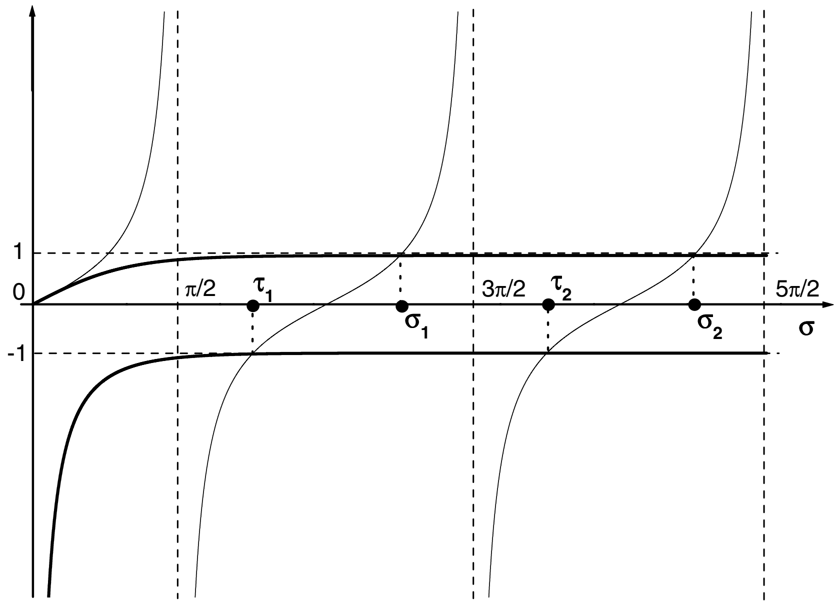

Figure 3. Graphical solutions $s_n = (\sigma_n, \tau_n)$ of equations (62) with $\eta = 1$ in the σ-space shown by dots along the σ-axis.

Note that for small σ equation (60) reduces to equation (47). At those values of σ which satisfy equation (60), we have the limits $W, \Delta_2 \to 0$. Next, from equation (26), in the $l \to 0$ limit, we obtain (cf equation (49)) the following asymptotics,

$$\Delta_1 \to \cosh(2\sigma)\cos(2\sqrt{\eta}\sigma) + \frac{1}{2}\left(\sqrt{\eta} - \frac{1}{\sqrt{\eta}}\right)\sinh(2\sigma)\sin(2\sqrt{\eta}\sigma), \qquad (61)$$

valid for all σ without constraint (60).

Similarly to equation (47), constraint (60) can also be rewritten as a quadratic equation with respect to $\tan(\sqrt{\eta}\sigma)$, leading to the two equations:

$$\tan(\sqrt{\eta}\sigma) = \sqrt{\eta}\tanh\sigma \qquad \text{and} \qquad \tan(\sqrt{\eta}\sigma) = -\coth\sigma/\sqrt{\eta}. \qquad (62)$$

Again, the first of these equations coincides with the corresponding condition for resonant tunnelling through the barrier–well rectangular system [33]. The second equation describes new resonances, which appear due to the presence of the second barrier in the BWB system. As illustrated by figure 3, the set of (non-zero) roots of the first equation $\{\sigma_n\}_{n=1}^{\infty}$ is located on intervals (52), whereas the second series of roots $\{\tau_n\}_{n=1}^{\infty}$ lies on intervals (53).

Similarly, we denote each pair of the resonance points σ_n and τ_n by s_n. Then, inserting equations (62) into the rhs of (61), we find that these asymptotics reduce to $\Delta_1 \to \pm 1$, where the upper sign belongs to the resonances at $\{\sigma_n\}_{n=1}^{\infty}$ and the lower one for $\{\tau_n\}_{n=1}^{\infty}$. Thus, as follows from equations (22), at the resonances in the σ-space $\{s_n\}_{n=1}^{\infty}$ given by equations (62), we again obtain the limits (55). In other words, for the point $\mu = \nu = 2$ we have full resonant tunnelling, in contrast to partial resonant tunnelling for the δ'-interaction [33].

4. Scattering properties of the WBW structure

Similarly, one can investigate the potentials with opposite sign, which correspond to the lower sign in the first equation (18). In order to keep the most of calculations from the previous section, for regularization of the potential $-\delta''(x)$ it is convenient to use the structure consisting of a barrier of height h and width $2l$ surrounded by two identical wells of depth d and width r:

$$V_{r,2l,r}(x) \doteq \lambda \begin{cases} 0 & \text{for} \quad -\infty < x < -l-r \\ -d & \text{for} \quad -l-r < x < -l \\ h & \text{for} \quad -l < x < l \\ -d & \text{for} \quad l < x < l+r \\ 0 & \text{for} \quad l+r < x < \infty. \end{cases} \tag{63}$$

Keeping the same notations (21), in the similar way we obtain that Δ_1 in (22) is given by the same equation (26), whereas W and Δ_2 have the following slightly modified form:

$$W \doteq \frac{\lambda}{k} D_1 \cosh^2(pl)\cos^2(qr), \tag{64}$$

$$D_1 \doteq \frac{h}{p}\tanh(pl) - \frac{d}{q}[1+\tanh^2(pl)]\tan(qr) + \frac{1}{p}\left[d + (h+d)\frac{k^2}{q^2}\right]\tanh(pl)\tan^2(qr), \tag{65}$$

$$\Delta_2 \doteq D_2 \cosh(2pl)\cos^2(qr), \tag{66}$$

$$D_2 \doteq \frac{1}{2}\left(\frac{p}{k} - \frac{k}{p}\right)\tanh(pl) - \left(\frac{q}{k} + \frac{k}{q}\right)\tan(qr) + \frac{1}{2}\left(\frac{q^2}{kp} - \frac{kp}{q^2}\right)\tanh(2pl)\tan^2(qr). \tag{67}$$

In contrast to the symmetric BWB potential (19), it is not evident that the full resonant tunnelling regime can exist for the WBW structure given by equations (63). Therefore we need first to analyse the possibility of resonances for a regularized potential (63).

4.1. The condition for resonances in a finite WBW structure

For the existence of resonances with full transparency through system (63) the equality $D_1 = 0$ (see equation (64)) has to be accomplished. According to equation (65), this equality takes the following explicit form:

$$\left[1 + \left(1 + \frac{h}{d}\right)\frac{k^2}{q^2}\right]\tan(qr) + \frac{h}{d}\cot(qr) = \frac{p}{q}[\tanh(pl) + \coth(pl)]. \tag{68}$$

Equation (68) can be rewritten as a quadratic equation with respect to $\tan(qr)$. The solution of the resulting quadratic equation can be written in the form

$$\tan\left(qr_n^{\pm}\right) = \frac{p}{2q[1+(1+h/d)k^2/q^2]}\left[\tanh(pl) + \coth(pl) \pm \sqrt{[\tanh(pl) - \coth(pl)]^2 - 4(1+h/d)^2k^2/p^2}\right] \tag{69}$$

with two series of roots $\{r_n^{\pm}\}_{n=0}^{\infty}$. The necessary condition for the existence of these roots is the inequality

$$2(1+h/d)k/p \leqslant \coth(pl) - \tanh(pl). \tag{70}$$

Since the rhs of equation (69) is positive, each pair of the roots $r_n^{\pm}$ lies on the interval

$$n\pi q^{-1} < r_n^{\pm} < (n+1/2)\pi q^{-1}, \qquad n = 0, 1, \ldots. \tag{71}$$

As follows from (70), the fully transparent regime can exist for a wide family of rectangular WBW structures, but the existence of both the wells appears to be crucial for resonant tunnelling. Indeed, in the $d \to 0$ limit inequality (70) obviously fails. On the other hand, this inequality is always satisfied in the zero-range limit. Indeed, if $pl \to 0$, using equations (32), one obtains $hl/d = \mathcal{O}(l^{1-\mu+\nu}) \to 0$ for the region Ω_p defined by (34). In the case $pl \to$ const we have $h/d = \eta$, $p = \mathcal{O}(l^{-1})$ and again inequality (70) holds in the $l \to 0$ limit.

In the case of equality in (70) each pair of the roots degenerates, so that $r_n^{\pm} \to r_n^0$. From this equality one can find the equation

$$\tanh(pl) + \coth(pl) = 2\sqrt{1 + (1 + h/d)^2 k^2/p^2}. \tag{72}$$

Using equations (21), from (69) we find the limits

$$\tan\left(qr_n^{\pm}\right) \to \tan\left(qr_n^0\right) = \frac{p}{2q} \cdot \frac{\tanh(pl) + \coth(pl)}{1 + (1 + h/d)k^2/q^2}. \tag{73}$$

Using next relation (72), the last equation can be simplified to

$$\tan\left(qr_n^0\right) = \sqrt{\frac{h}{d + (h + d)k^2/q^2}}, \qquad n = 0, 1, \ldots. \tag{74}$$

4.2. The zero-range limit of the WBW potential

In a similar way as in the previous section, for case (i), i.e., on the set Ω_0, we obtain the same asymptotics (41) in the zero-range limit, leading to the same results: the existence of the fully transparent regime on the set Ω_f and the effective δ-interaction on the line L_1 described by the scattering amplitudes (43) with the effective coupling constant (44).

For case (ii), when $pl \to 0$ but $qr \to$ const, we also obtain a constraint at which the transparency turns out to be non-zero. This constraint takes the form of the equation

$$\sin(2\sqrt{\eta}\sigma) = 2\sqrt{\eta}\sigma \cos^2(\sqrt{\eta}\sigma), \tag{75}$$

which, similarly to equation (47), reduces to the following two equations:

$$\tan(\sqrt{\eta}\sigma) = \sqrt{\eta}\sigma \qquad \text{and} \qquad \cos(\sqrt{\eta}\sigma) = 0. \tag{76}$$

Again, the first of these equations coincides with the first equation (51) and therefore it describes the same spectrum of resonances given by the points $\{\bar{\sigma}_n\}_{n=1}^{\infty}$ in the σ-space. The second equation (76) differs from that in (51) and therefore the resonances with the solutions $\bar{\tau}_n = (n - 1/2)\pi\eta^{-1/2}$, $n = 1, 2, \ldots$, are specific for the $-\delta''$-like point interaction.

In spite of the difference of the points $\{\bar{\tau}_n\}_{n=1}^{\infty}$ given by the different second equations (51) and (76), the asymptotics for Δ_1 at the resonance points appear to be the same, namely, $\Delta_1 \to \pm 1$, where again the upper sign belongs to the resonances at the points $\{\bar{\sigma}_n\}_{n=1}^{\infty}$ and the lower one to the points $\{\bar{\tau}_n\}_{n=1}^{\infty}$. We also find that $W, \Delta_2 \to \infty$ on the set Ω_4, except for the values $\bar{s}_n = (\bar{\sigma}_n, \bar{\tau}_n)$, $n = 1, 2, \ldots$, where both W and Δ_2 tend to zero.

At the limiting point Ω_5 we have the asymptotics

$$W, \Delta_2 \to \begin{cases} -\bar{\sigma}_n^4/3k\left(1 + \eta\bar{\sigma}_n^2\right) & \text{for} \quad \sigma = \bar{\sigma}_n \\ \bar{\tau}_n^2/k\eta & \text{for} \quad \sigma = \bar{\tau}_n, \qquad n = 1, 2, \ldots, \end{cases} \tag{77}$$

inserting which into equations (22) we obtain the scattering amplitudes (56), but with the renormalized coupling constant g given by

$$g = \begin{cases} -2\bar{\sigma}_n^4/3k\left(1 + \eta\bar{\sigma}_n^2\right) & \text{for} \quad \sigma = \bar{\sigma}_n \\ -2\bar{\tau}_n^2/k\eta & \text{for} \quad \sigma = \bar{\tau}_n, \quad n = 1, 2, \ldots. \end{cases} \tag{78}$$

Similarly, at the point Ω_6 we obtain the following condition for resonances (with $W, \Delta_2 \to 0$),

$$\eta + \tan^2(\sqrt{\eta}\sigma) = 2\sqrt{\eta}\coth(2\sigma)\tan(\sqrt{\eta}\sigma), \tag{79}$$

which splits into the two simple equations

$$\tan(\sqrt{\eta}\sigma) = \sqrt{\eta}\tanh\sigma \qquad \text{and} \qquad \tan(\sqrt{\eta}\sigma) = \sqrt{\eta}\coth\sigma. \tag{80}$$

Again, the first of these equations coincides with that for the δ'-interaction [33], whereas the second one is specific for the WBW structure. For the solutions of both equations (80) we keep the same notations, namely, $\{\sigma_n, \tau_n\}_{n=1}^{\infty}$. In the same way, from (61) we obtain the $l \to 0$ limits $\Delta_1 \to \pm 1$, where the upper sign corresponds to the first and the lower one to the second equation in (80). Thus, for the resonance points $\sigma = \sigma_n, \tau_n, n = 1, 2, \ldots,$ we have full transmission through the $-\delta''$-like point potential with the scattering amplitudes (55). For the values $\sigma \neq \sigma_n$ or $\tau_n, n = 1, 2, \ldots,$ the half-lines $\mathbb{R}^{\pm}$ are separated ($R = -1$ and $T = 0$).

5. Boundary conditions on wavefunctions in the zero-range limit

In this section we consider the boundary conditions on the wavefunction $\psi(x)$ at $x = 0$ in the zero-range limit (when $l, r \to 0$). From the finite-range equations (20), one can write the following equations (with accuracy to an arbitrary constant, being the same for all boundary conditions) for the left (at $x = -l - r \to -0$) and the right (at $x = l + r \to +0$) boundary values of the wavefunction $\psi(x)$ and its derivative $\psi'(x)$:

$$\begin{aligned} &\psi(-0) = 1 + R, \qquad &&\psi(+0) = T, \\ &\psi'(-0) = \mathrm{i}k(1 - R), \qquad &&\psi'(+0) = \mathrm{i}kT. \end{aligned} \tag{81}$$

In the case of full transmission ($R \to 0$ and $T \to 1$), which takes place on the set Ω_f, equations (81) become $\psi(-0) = \psi(+0) = 1$, $\psi'(-0) = \psi'(+0) = \mathrm{i}k$. Therefore the matrix equation (2) is fulfilled if $\Lambda = I$. Here both the wavefunction $\psi(x)$ and its derivative $\psi'(x)$ are continuous at $x = 0$ and therefore the corresponding point interaction is trivial.

Similarly, in the case of resonances with $R \to 0$ and $T \to \pm 1$, which occur on the sets Ω_4 and Ω_6, equations (81) reduce to $\psi(-0) = 1$, $\psi(+0) = \pm 1$, $\psi'(-0) = \mathrm{i}k$, $\psi'(+0) = \pm \mathrm{i}k$ and therefore in equation (2) we have $\Lambda = \pm I$. Here we are dealing with the case of full resonant tunnelling through a $\pm\delta''$-like point interaction. For other values σ, we have $R = -1$ and $T = 0$, i.e., the half-lines $\mathbb{R}^{\pm}$ are separated.

One can use the general relations for R and T, derived by Cheon *et al* [31], which follow from equations (4)–(7) and (81):

$$R = \frac{\alpha Q + \alpha^* Q^{-1} - (\mathrm{e}^{\mathrm{i}\xi} + \mathrm{e}^{-\mathrm{i}\xi})}{\mathrm{e}^{\mathrm{i}\xi} Q + \mathrm{e}^{-\mathrm{i}\xi} Q^{-1} - (\alpha + \alpha^*)}, \qquad T = -\frac{\beta(Q - Q^{-1})}{\mathrm{e}^{\mathrm{i}\xi} Q + \mathrm{e}^{-\mathrm{i}\xi} Q^{-1} - (\alpha + \alpha^*)}, \tag{82}$$

where

$$Q \doteq \frac{1 - kL_0}{1 + kL_0}. \tag{83}$$

For the non-separating particular cases with $R = 0$ and $T = \pm 1$, it follows immediately from the first equation (82) that $\alpha = 0, \xi = \pi/2$, while imposing $T = \pm 1$ in the second equation (82), we obtain $\beta = \mp \mathrm{i}$.

Consider now the case of partial tunnelling which occurs on the line L_1 as well as at the point Ω_5. First consider the line L_1, where the transmission is given by equations (43) with (44). Inserting these values for R and T into equations (81), we obtain the same boundary values as previously [33] found for the δ'-interaction, namely

$$\psi(-0) = \psi(+0) = 1, \qquad \psi'(-0) = \mathrm{i}k - g, \qquad \psi'(+0) = \mathrm{i}k \tag{84}$$

where the constant g is given by equations (44). Then the matrix equation (2) is fulfilled with the coefficients (11). In this case we are dealing with the same kind of transmission when the system effectively behaves as the δ-interaction (10), but with the effective coupling constant g being twice bigger than for the corresponding δ'-interaction [33].

In a similar way, using the scattering amplitudes (56), we find

$$\psi(-0) = 1, \qquad \psi(+0) = \pm 1, \qquad \psi'(-0) = \mathrm{i}k - g, \qquad \psi'(+0) = \pm \mathrm{i}k. \tag{85}$$

In this case the matrix equation (2) will be fulfilled if we put

$$\lambda_{11} = \lambda_{22} = \pm 1, \qquad \lambda_{12} = 0, \qquad \lambda_{21} = \pm g, \qquad \chi = 0. \tag{86}$$

This case corresponds to partial resonant tunnelling through the renormalized $\pm\delta''$-potential.

Note that the finite values for R and T, which correspond to the δ-interaction, can directly be obtained from the general formulae (82), if we put $\alpha_R = -\cos\xi$ and $\alpha_I = 0$. Next, in order to obtain the scattering amplitudes (56), it is sufficient to put $\beta = \mp\mathrm{i}\sin\xi$ and $\tan\xi = 2/L_0 g$. Therefore the U matrix for the effective δ-interaction, which appears on the line L_1 for any value of the coupling constant λ and as resonant tunnelling (for a discrete set in the λ-space) at the point Ω_5, has the form

$$U = \mathrm{e}^{\mathrm{i}\xi} \begin{pmatrix} -\cos\xi & \mp\mathrm{i}\sin\xi \\ \mp\mathrm{i}\sin\xi & -\cos\xi \end{pmatrix}, \tag{87}$$

where the parameter ξ is given by equation (13).

Finally, we note that in the region $\Omega_p \backslash (\Omega_f \cup L_1 \cup L_2 \cup \Omega_6)$, outside the discrete sets of resonance points in the σ-space, which occur on the set $L_2 \cup \Omega_6$, we have separated point interactions with $W, \Delta_2 \to \infty$. In the $l \to 0$ limit equations (22) imply $R = -1$ and $T = 0$. Therefore for this type of separated point interactions equations (82) result in the U matrix with $\alpha = -\mathrm{e}^{\mathrm{i}\xi}$ and $\beta = 0$ with any $\xi \in [0, \pi)$.

6. Conclusions

In this paper we have analysed a family of point interactions with non-trivial scattering properties exhibiting full and partial resonant tunnelling. The construction of zero-range singular potentials of this type, which we refer to as $\pm\delta''$-like point interactions, involves explicitly three arbitrary positive parameters. One of these parameters (η) controls the rate of squeezing the well (barrier) compared to squeezing both the barriers (wells). The others two describe the rate of increasing the barrier height (μ) and the well depth (ν). As a result, the three-parameter family of $\pm\delta''$-like point interactions has been constructed and analysed in detail. In particular, on the $\{\mu, \nu\}$-plane, the regions of point interactions with non-trivial scattering properties have been found.

In general, under the description of all possible point interactions arising from the zero-range limit, it is assumed that any interaction located at $x = 0$ is considered as two interacting subsystems lying on the half-axes $\mathbb{R}^{\pm}$. As shown in the present paper, among these limiting cases there are those for which the interaction between the subsystems results in full transparency. The family of this type is given by all pairs $\{\mu, \nu\} \in \Omega_f$ illustrated by figure 1.

The interesting families of point interactions are found on the two lines L_1, L_2 and the isolated point Ω_6. The line L_1, which splits all the point interactions into fully transparent and non-transparent potentials, corresponds to the renormalized interaction $\pm\lambda\delta''(x)$ that effectively does not differ from the point interaction $g\delta(x)$, where the constant g is given in terms of the coupling constant λ or σ (see equation (36)) through equation (44). Since in this case the $\pm\delta''$-like potential can be considered as two attached each to other δ'-like systems, the effective coupling constant g appears to be twice bigger than the corresponding constant for the

δ'-like interactions located along the same line L_1. The second line L_2 and the isolated point Ω_6 contain a family of non-trivial point interactions with resonant tunnelling. The resonances appear at some fixed values of the coupling constant λ being countable sets in the λ-space. On the sets Ω_4 and Ω_6 the resonance conditions given by equations (51), (62), (76) and (80) provide full transparency, while at the limiting point Ω_5 we have partial resonant transmission given by amplitudes (56) with the coupling constants (57) and (78) for the BWB and the WBW systems, respectively. At these resonances the system behaves effectively as a δ-well potential.

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